Design Techniques for Uniform-DFT, Linear Phase Filter Banks

Honglin Sun and Phillip De Leon*

* Direct all correspondence regarding this manuscript to Dr. Phillip De Leon.

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Abstract

Uniform-DFT filter banks are an important class of filter banks and their theory is well known. One notable characteristic is their very efficient implementation when using polyphase filters and the FFT. Separately, linear phase filter banks, i.e. filter banks in which the analysis filters have a linear phase are also an important class of filter banks and desired in many applications. Unfortunately, it has been proved that one cannot design critically-sampled, uniform-DFT, linear phase filter banks and achieve perfect reconstruction. In this paper, we present a least-squares solution to this problem and in addition prove that oversampled, uniform-DFT, linear phase filter banks (which are also useful in many applications) can be constructed for perfect reconstruction. Design examples are included which illustrate the methods.

Keywords

Uniform-DFT filter banks, Linear phase filter banks, Oversampled filter banks.

I. Introduction

In the last decade research in filter banks and subband processing has occurred at a fast pace [12], [16], [17]. In particular, subband processing of signals is now common in a number of applications such as audio, image, and video encoders/decoders; acoustic echo cancelers used in teleconferencing systems; and spread spectrum communications systems [1]. In addition, emerging applications include multiple target tracking in radar, high-data-rate digital receivers for satellite channels, and suppression of interference signals in wireless applications [7], [13], [6], [2]. As compared to equivalent fullband processing, subband signal processing in these applications often leads to a performance increase due to the frequency band decomposition and potentially a reduction in computation due to the lower sampling rate of the subband (component) signals.

An analysis bank decomposes (analyzes) a signal, x by partitioning its spectrum into frequency bands or subbands by using a set of M bandpass filters (BPFs) denoted $\mathbf{h}_0, ..., \mathbf{h}_{M-1}$ in Fig. 1. Once partitioned, the signals are then downsampled by a factor of D (due to their reduced bandwidth). Upon analysis, each of these subband signals $(x_0, ..., x_{M-1})$ may be processed (subband processing). After processing, the subband signals are synthesized into the fullband signal, $\hat{x}$, by first upsampling (zero insertion) to the original sampling rate followed by bandpass filtering $(\mathbf{f}_0, ..., \mathbf{f}_{M-1})$ to remove spectral images introduced from upsampling. Such a system of upsamplers and bandpass (synthesis) filters is referred to as a synthesis bank. The combination of analysis and synthesis banks is called a filter bank [17].

II. UNIFORM-DFT FILTER BANKS

One of the more popular designs for filter banks, called the uniform-DFT filter bank, assumes that all analysis, synthesis filters are frequency-shifted versions of an analysis, synthesis prototype lowpass filter (LPF), $\mathbf{h}_0$, $\mathbf{f}_0$ respectively [17]. The time-domain relations are then given by

$$
\begin{aligned}
h_m(n) &= h_0(n)e^{j2\pi m[n-(N-1)/2]/M} \\
f_m(n) &= f_0(n)e^{j2\pi m[n-(N-1)/2]/M}
\end{aligned}
\tag{1}
$$

where $m = 1, \ldots, M-1$ denotes the subband and N is the length of the filter. The frequency-domain relations are then given by

$$
\begin{aligned}
H_m\left(e^{j\omega}\right) &= H_0\left(e^{j(\omega-2\pi m/M)}\right)e^{-j\pi m(N-1)/M} \\
F_m\left(e^{j\omega}\right) &= F_0\left(e^{j(\omega-2\pi m/M)}\right)e^{-j\pi m(N-1)/M}.
\end{aligned}
\tag{2}
$$

A typical magnitude response of analysis/synthesis filters related by (2) with $M = 4$ is given in Fig. 2.

A. Polyphase, Uniform-DFT Filter Banks

In the analysis bank shown in Fig. 1, much computation is wasted in the implementation since after filtering, only one out of every D samples is retained. Similar filtering inefficiencies occur in the synthesis bank since only one out of every D samples is non-zero prior to synthesis filtering. A more efficient but equivalent implementation, called a polyphase, uniform-DFT filter bank relies on a polyphase representation of $\mathbf{h}_0$ and $\mathbf{f}_0$ (prototype LPFs) and a discrete Fourier transform (DFT) and inverse DFT (IDFT) as shown in Fig. 3 [12], [17]. Note that in Fig. 3, $I = M/D$ is referred to as the *oversampling* factor and will be discussed in the next section. The polyphase filter coefficients used in Fig. 3 are given by

$$
\begin{aligned}
e_m(n) &= h_0(nD - m) \\
r_m(n) &= f_0(nD - m)
\end{aligned}
\tag{3}
$$

where $m = 0, \ldots, M-1$. As an example, if we assume $M = 4$, $D = 2$, and a causal, FIR analysis filter prototype, $\mathbf{h}_0$ the polyphase filters would be given by

$$
\begin{aligned}
\mathbf{e}_0 &= [h_0(0), h_0(2), h_0(4), h_0(6), \ldots]^T \\
\mathbf{e}_1 &= [0, h_0(1), h_0(3), h_0(5), \ldots]^T \\
\mathbf{e}_2 &= [0, h_0(0), h_0(2), h_0(4), \ldots]^T \\
\mathbf{e}_3 &= [0, 0, h_0(1), h_0(3), \ldots]^T.
\end{aligned}
\tag{4}
$$

With the polyphase representation, analysis and synthesis filtering occurs at the lower, subband sampling rate resulting in a lower processing speed. In addition, the DFT and IDFT are implemented with a fast-Fourier transform (FFT). These combined effects result in a significant computational reduction.

In most of the literature, filter bank design is aimed at perfectly reconstructing x (to within a scale factor and delay) from the critically-sampled ($D = M$ or $I = 1$) subband signals $x_0, ..., x_{M-1}$ (Fig. 3). Critically-sampled refers to the fact that the number of fullband samples per second equals the total number of subband samples per second. In this case, one must carefully control the aliasing in the subband signals through proper design of the analysis and synthesis filters. Many critically-sampled designs are available that yield the perfect reconstruction property [15].

B. Oversampled, Polyphase, Uniform-DFT Filter Banks

A fundamental problem in processing critically-sampled subband signals is the large level of subband aliasing which is present regardless of the analysis and synthesis filter designs due to the overlapping filters. With such a large amount of aliasing (which acts as a disruptive noise signal), many subband signal processing applications do not perform well and may require a large additional overhead to partially compensate for the aliasing [5].

One solution to the subband aliasing problem in applications is to oversample ($D < M$) the subband signals to minimize the effect of the aliasing [12], [3]. We typically choose the oversampling factor to be close to unity in order to minimize the data rate or choose $I = 2$ for simple implementation as in Fig. 3. In the latter case, this implies twice the total number of samples per second in the subbands than are required at the fullband rate. Howver, even with oversampled subbands, many signal processing applications can have fewer total computations per sample as compared to the fullband case [4].

III. Linear Phase Filter Banks

Linear phase filter banks are designed so that the analysis filters have a linear phase response. This should not be confused with the idea that the frequency response of the filter bank has linear phase, i.e. that the end-to-end impulse response of the filter bank is symmetric. There are many filter bank applications that require linear phase analysis filters such as subband image coders and filter bank-based digital receivers [13], [6], [18]. The basic theory of linear phase filter banks with perfect reconstruction is also well-known (for complete details, the reader is referred to [11]).

IV. UNIFORM-DFT, LINEAR PHASE FILTER BANKS

It has been shown that critically-sampled uniform-DFT filter banks in which the analysis filters have linear phase cannot be designed to have perfect reconstruction [8], [9]. In this section, we analyze the uniform-DFT, linear phase filter bank and formulate a least-squares synthesis filter solution in the critically-sampled case which approximates perfect reconstruction. In the oversampled case, we formulate a synthesis filter design that leads to perfect reconstruction.

A. Time-Domain Analysis

In order to formulate the design for the uniform-DFT linear phase filter bank, we begin by characterizing the family of responses to a unit-pulse with delay, l or $\delta(n-l)$ for $1 \leq l \leq D$. In general, the filter bank is not time-invariant but rather periodically time-invariant with period D thus resulting in our need to consider this family. The subband signals (Fig. 1) in response to $\delta(n-l)$ can be written as

$$x_{m,l}(n) = h_m(nD - l). \tag{5}$$

After upsampling and synthesis filtering the subband signals in (5), we have

$$y_{m,l}(n) = \sum_k h_m(kD - l) f_m(n - kD) \tag{6}$$

as in Fig. 1. By substituting the prototype filter relations in (1) into (6) we have

$$y_{m,l}(n) = \sum_k h_0(kD - l) f_0(n - kD) e^{j2\pi m[n-l-(N-1)]/M}. \tag{7}$$

The response of the filter bank to $\delta(n-l)$ is then sum of the signals in (7) or

$$
\begin{aligned}
\hat{x}_l(n) &= \sum_{m=0}^{M-1} y_{m,l}(n) \\
&= \sum_k h_0(kD - l) f_0(n - kD) \frac{1 - e^{j2\pi[n-l-(N-1)]}}{1 - e^{j2\pi[n-l-(N-1)]/M}} \\
&= M \sum_q \sum_k h_0(kD - l) f_0[l + (N - 1) - kD - qM] \delta[n - l - (N - 1) + qM] \tag{8}
\end{aligned}
$$

since

$$\frac{1 - e^{j2\pi[n-l-(N-1)]}}{1 - e^{j2\pi[n-l-(N-1)]/M}} = \sum_q M\delta[n - l - (N - 1) + qM]. \tag{9}$$

One condition for perfect reconstruction (with delay) is that the family of responses to the delayed unit-pulse is equal to a family of equivalently delayed unit-pulses. Specifically we require

$$\hat{x}_l(n) = \delta[n - l - (N - 1)] \tag{10}$$

for $1 \leq l \leq D$.

We can arrange the conditions in (10) using (8) in matrix form as

$$\mathbf{H}\mathbf{f}_0 = \frac{1}{M}\Delta \tag{11}$$

where

$$\mathbf{H} = \begin{bmatrix} 0 & h_0(D) & 0 & h_0(0) & \cdots & 0 & 0 & 0 \\ h_0(D+1) & 0 & h_0(1) & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & h_0(q_{\max,1}D) & 0 & h_0(q_{\max,1}D - D) & \cdots & 0 & 0 & h_0(0) \\ h_0(q_{\max,2}D + D - 1) & 0 & h_0(q_{\max,2}D - 1) & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & h_0(q_{\max,D-1}D + 2) \\ 0 & 0 & 0 & 0 & \cdots & h_0(q_{\max,D}D + 1) & 0 & 0 \end{bmatrix} \tag{12}$$

with dimension

$$\dim(\mathbf{H}) = \left[\sum_{l=1}^{D} \left(2 \left\lfloor \frac{N-1+l}{M} \right\rfloor - 1 \right) \right] \times N, \tag{13}$$

$$q_{\max,l} = \left\lfloor \frac{N-1+l}{D} \right\rfloor, \tag{14}$$

and

$$\Delta = \begin{bmatrix} \underbrace{0 \quad \cdots \quad 0}_{0} & \underbrace{1}_{\sum_{l=1}^{D}\lfloor\frac{N-1+l}{M}\rfloor - D} & \cdots & \underbrace{1}_{\sum_{l=1}^{D}\lfloor\frac{N-1+l}{M}\rfloor - 1} & 0 \quad \cdots \quad 0 \end{bmatrix}^{T}. \tag{15}$$

B. Critically-Sampled Case

In the critically-sampled case, $D = M$ we consider the matrix form of (10) for each l individually. This is equivalent to decomposing the matrix equation in (11) into a series of matrix equations [due to the decoupled nature of (11)] as

$$\bar{\mathbf{H}}_l \bar{\mathbf{f}}_l = \frac{1}{M}\delta_l \tag{16}$$

for $1 \leq l \leq D$ where

$$\bar{\mathbf{H}}_l = \begin{bmatrix} h_0(M-l) & 0 & \cdots & 0 & 0 \\ h_0(2M-l) & h_0(M-l) & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ h_0(q_{\max,l}M - l) & h_0[(q_{\max,l}-1)M - l] & \cdots & h_0(2M-l) & h_0(M-l) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & h_0(q_{\max,l}M - l) & h_0[(q_{\max,l}-1)M - l] \\ 0 & 0 & 0 & 0 & h_0(q_{\max,l}M - l) \end{bmatrix} \tag{17}$$

$$
\bar{\mathbf{f}}_l \;=\; \begin{bmatrix} f_0\left[(N-1)+l-q_{\max,l}M\right] \\[4pt] f_0\left[(N-1)+l-(q_{\max,l}-1)M\right] \\[4pt] \vdots \\[4pt] f_0\left[(N-1)+l-2M\right] \\[4pt] f_0\left[(N-1)+l-M\right] \end{bmatrix}, \tag{18}
$$

and

$$
\boldsymbol{\delta}_l \;=\; \begin{bmatrix} 0 & \cdots & 0 & \underbrace{1}_{\lfloor\frac{N-1+l}{M}\rfloor} & 0 & \cdots & 0 \end{bmatrix}^T. \tag{19}
$$

Note that only a subset of the synthesis prototype coefficients are present in (18). However, over the full range of l all coefficients are accounted for.

By choosing the filter length, $N = pM$ for integer p, the matrix $\bar{\mathbf{H}}_l$ is $(2p-1) \times p$ and $\bar{\mathbf{f}}_l$ is $p \times 1$. For this case, our problem is overdetermined and as well known, there is no solution for the set of $\bar{\mathbf{f}}_1, \bar{\mathbf{f}}_2 \ldots, \bar{\mathbf{f}}_M$ that constitute the synthesis prototype filter, $\mathbf{f}_0$ and leads to perfect reconstruction [9]. However, we can formulate a new, least-squares (LS) design problem as follows.

Given: $\bar{\mathbf{H}}_l \in \Re^{(2p-1)\times p}, \boldsymbol{\delta}_l \in \Re^{(2p-1)\times 1}$ for $1 \le l \le M$

Minimize: $V(\bar{\mathbf{f}}_l) = \|\boldsymbol{\delta}_l - \bar{\mathbf{H}}_l \bar{\mathbf{f}}_l\|^2$ for $1 \le l \le M$.

The LS solution to this problem is

$$
\bar{\mathbf{f}}_l \;=\; \bar{\mathbf{H}}_l^{+} \boldsymbol{\delta}_l \tag{20}
$$

where

$$
\bar{\mathbf{H}}_l^{+} \;=\; \left(\bar{\mathbf{H}}_l^{*}\bar{\mathbf{H}}_l\right)^{-1}\bar{\mathbf{H}}_l^{*} \tag{21}
$$

is the pseudoinverse of $\bar{\mathbf{H}}_l$, $\det(\bar{\mathbf{H}}_l^{*}\bar{\mathbf{H}}_l) \neq 0$, and $\bar{\mathbf{H}}_l^{*}$ is the complex conjugate transpose of $\bar{\mathbf{H}}_l$.

For the choice $N = pM + q$ where $1 \le q \le M - 1$, $\bar{\mathbf{f}}_l$ will have a different length according to l but the problem remains overdetermined and the least-squares solution in (20) still holds. General LS approximations of perfect reconstruction filter banks are discussed in [14].

By design, the analysis prototype LPF, $\mathbf{h}_0$ has a linear phase and symmetric impulse response of the form

$$
\mathbf{h}_0(n) \;=\; \mathbf{h}_0\left[(N-1)-n\right] \tag{22}
$$

where $n = 0, \ldots, N - 1$. From (1) it is clear that the analysis filters, $\mathbf{h}_1, \ldots, \mathbf{h}_{M-1}$ will also have linear phase responses. We can prove that under the formulation in (11), the LS solution for the synthesis

prototype LPF $\mathbf{f}_0$ will also have a linear phase and hence the synthesis filters $\mathbf{f}_1, \ldots, \mathbf{f}_{M-1}$ will also have linear phase responses. The proof is as follows.

Consider the case where $N = pM$. We have $q_{max} = p$ from (14), and applying (22) to (17) and substituting the result into (16) we have

$$
\begin{bmatrix}
h_0\left[(N-1)+l-M\right] & 0 & \cdots & & 0 \\
h_0\left[(N-1)+l-2M\right] & \ddots & \ddots & & \vdots \\
\vdots & \ddots & \ddots & & 0 \\
h_0(l-1) & & \ddots & h_0\left[(N-1)+l-M\right] \\
0 & \ddots & \ddots & h_0\left[(N-1)+l-2M\right] \\
\vdots & & \ddots & & \vdots \\
0 & \cdots & 0 & & h_0(l-1)
\end{bmatrix}
\begin{bmatrix}
f_0(l-1) \\
f_0(l-1+M) \\
\vdots \\
f_0\left[(N-1)+l-2M\right] \\
f_0\left[(N-1)+l-M\right]
\end{bmatrix}
=
\begin{bmatrix}
0 \\
\vdots \\
0 \\
1 \\
0 \\
\vdots \\
0
\end{bmatrix}
\quad (23)
$$

Now consider the vector

$$
\mathbf{f}_l' =
\begin{bmatrix}
f_0\left(M-l\right) \\
f_0\left(2M-l\right) \\
\vdots \\
f_0\left(N-M-l\right) \\
f_0\left(N-l\right)
\end{bmatrix}
\quad (24)
$$

and the equivalent to (23) which uses $\mathbf{f}_l'$ instead of $\bar{\mathbf{f}}_l$

$$
\begin{bmatrix}
h_0(N-l) & 0 & \cdots & & 0 \\
h_0(N-M-l) & \ddots & \ddots & & \vdots \\
\vdots & \ddots & \ddots & & 0 \\
h_0(M-l) & & \ddots & h_0(N-l) \\
0 & \ddots & \ddots & h_0(N-M-l) \\
\vdots & & \ddots & & \vdots \\
0 & \cdots & 0 & & h_0(M-l)
\end{bmatrix}
\begin{bmatrix}
f_0(M-l) \\
f_0(2M-l) \\
\vdots \\
f_0\left(N-M-l\right) \\
f_0\left(N-l\right)
\end{bmatrix}
=
\begin{bmatrix}
0 \\
\vdots \\
0 \\
1 \\
0 \\
\vdots \\
0
\end{bmatrix} .
\quad (25)
$$

Equation (25) is identical to (16) except that elements of the matrix along any diagonal are in reverse

order. We can rewrite (25) as

$$
\begin{bmatrix}
h_0(M-l) & 0 & \cdots & 0 \\
h_0(2M-l) & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
h_0(N-l) & & \ddots & h_0(M-l) \\
0 & \ddots & \ddots & h_0(2M-l) \\
\vdots & & \ddots & \vdots \\
0 & \cdots & 0 & h_0(N-1)
\end{bmatrix}
\begin{bmatrix}
f_0(N-l) \\
f_0(N-M-l) \\
\vdots \\
f_0(2M-l) \\
f_0(M-l)
\end{bmatrix}
=
\begin{bmatrix}
0 \\
\vdots \\
0 \\
1 \\
0 \\
\vdots \\
0
\end{bmatrix}.
\tag{26}
$$

Using the LS solution given in (20) in both (23) and (26) yields $\mathbf{f}_0$ that satisfies

$$
f_0(n) = f_0(N-1-n)
\tag{27}
$$

or in other words, $\mathbf{f}_0$ also has linear phase. The same result can also be shown for the case where $N = pM + q$ where $q = 1, \ldots, M-1$.

C. Oversampled Case with Integer Oversampling Factor

We next consider the oversampled case where the oversampling factor, $I \geq 2$ is an integer. In this case, we use (16) but with

$$
\bar{\mathbf{H}}_l =
\begin{bmatrix}
h_0(D-l) & 0 & \cdots & \cdots & 0 \\
h_0(2D-l) & h_0(D-l) & 0 & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
h_0(q_{\max,l}D-l) & h_0[(q_{\max,l}-1)D-l] & \cdots & h_0(2D-l) & h_0(D-l) \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 0 & h_0(q_{\max,l}D-l) & h_0[(q_{\max,l}-1)D-l] \\
0 & \cdots & \cdots & 0 & h_0(q_{\max,l}D-l)
\end{bmatrix}.
\tag{28}
$$

$$
\bar{\mathbf{f}}_l =
\begin{bmatrix}
f_0[(N-1)+l-q_{\max,l}D] \\
f_0[(N-1)+l-(q_{\max,l}-1)D] \\
\vdots \\
f_0[(N-1)+l-2D] \\
f_0[(N-1)+l-D]
\end{bmatrix}.
\tag{29}
$$

$l = 1, \ldots, D$, and $q_{\max,l}$ is given by (14).

By choosing $N = pM$ for integer p, the matrix $\bar{\mathbf{H}}_l$ is $(2p-1) \times Ip$ and $\bar{\mathbf{f}}_l$ is $Ip \times 1$. For this case, our problem is underdetermined and there exists an infinite number of solutions for $\bar{\mathbf{f}}_l$ and hence $\mathbf{f}_0$

that lead to perfect reconstruction. We therefore reformulate the design problem so as to choose the $\mathbf{f}_0$ that has minimum stopband energy. Assume the cutoff frequency of the analysis prototype filter is $\omega_c = \pi/M$. Then the synthesis prototype LPF we choose minimizes the cost function

$$J(\mathbf{f}_0) \;=\; \int_{\omega_c}^{\pi} |F_0(e^{j\omega})|^2 d\omega \tag{30}$$

where

$$F_0(e^{j\omega}) \;\equiv\; \sum_{n=0}^{N-1} f_0(n)e^{-j\omega n}. \tag{31}$$

Substituting (31) into (30) we have

$$J(\mathbf{f}_0) \;=\; \sum_{n=0}^{N-1}\left[f_0^2(n)\frac{(M-1)}{M}\pi \right] - \frac{2\pi}{M}\sum_{m=0}^{N-1} f_0(m)\sum_{k=1}^{N-1-m} f_0(m+k)\mathrm{sinc}(k/M)$$

$$\;=\; \frac{(M-1)\pi}{M}\mathbf{f}_0^T\mathbf{f}_0 - \frac{2\pi}{M}\mathbf{f}_0^T\mathbf{S}\mathbf{f}_0 \tag{32}$$

where $\mathrm{sinc}(x) \equiv \frac{\sin(\pi x)}{\pi x}$ and

$$\mathbf{S} \;=\; \begin{bmatrix} 0 & \frac{1}{2}\mathrm{sinc}(\frac{1}{M}) & \frac{1}{2}\mathrm{sinc}(\frac{2}{M}) & \cdots & \frac{1}{2}\mathrm{sinc}(\frac{N-1}{M}) \\ \frac{1}{2}\mathrm{sinc}(\frac{1}{M}) & 0 & \frac{1}{2}\mathrm{sinc}(\frac{1}{M}) & \ddots & \frac{1}{2}\mathrm{sinc}(\frac{N-2}{M}) \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \frac{1}{2}\mathrm{sinc}(\frac{N-1}{M}) & \frac{1}{2}\mathrm{sinc}(\frac{N-2}{M}) & \cdots & \frac{1}{2}\mathrm{sinc}(\frac{1}{M}) & 0 \end{bmatrix}. \tag{33}$$

We rewrite (32) as

$$J(\mathbf{f}_0) \;=\; \frac{(M-1)\pi}{M}\mathbf{f}_0^T\mathbf{U}\mathbf{f}_0 \tag{34}$$

where

$$\mathbf{U} = \mathbf{I} - (\frac{2}{M-1})\mathbf{S} \tag{35}$$

is a real. symmetric matrix. Since $J(\mathbf{f}_0) > 0$ for all $\mathbf{f}_0$, $\mathbf{U}$ is a positive definite matrix and there exists a non-singular matrix $\mathbf{C}$ that satisfies

$$\mathbf{C}^T\mathbf{U}\mathbf{C} \;=\; \begin{bmatrix} \lambda_0 & & \\ & \ddots & \\ & & \lambda_{N-1} \end{bmatrix} \tag{36}$$

where $\lambda_0, \cdots, \lambda_{N-1}$ are the positive eigenvalues of $\mathbf{U}$. Define

$$\tilde{\mathbf{f}}_0 \;\equiv\; \mathbf{P}\mathbf{f}_0 \tag{37}$$

where

$$P = \begin{bmatrix} \sqrt{\lambda_0} & & \\ & \ddots & \\ & & \sqrt{\lambda_{N-1}} \end{bmatrix} C^{-1}. \tag{38}$$

Using (36)-(38) in (34) we have

$$J(\mathbf{f}_0) = \frac{(M-1)\pi}{M} \tilde{\mathbf{f}}_0^T \tilde{\mathbf{f}}_0. \tag{39}$$

Substituting $\mathbf{f}_0 = \mathbf{P}^{-1}\tilde{\mathbf{f}}_0$ into (11) we have

$$\mathbf{H}\mathbf{P}^{-1}\tilde{\mathbf{f}}_0 = \frac{1}{M}\Delta \tag{40}$$

or

$$\tilde{\mathbf{H}}\tilde{\mathbf{f}}_0 = \frac{1}{M}\Delta \tag{41}$$

where

$$\tilde{\mathbf{H}} = \mathbf{H}\mathbf{P}^{-1} \tag{42}$$

and has dimension $(\frac{2N}{I} - D) \times N$. We can now formulate the design problem for the oversampled case (integer I) under the minimum stopband constraint.

Given: $\tilde{\mathbf{H}} \in \Re^{(\frac{2N}{I}-D)\times N}, \Delta \in \Re^{(\frac{2N}{I}-D)\times 1}$

Minimize: $V(\tilde{\mathbf{f}}_0) = \|\tilde{\mathbf{f}}_0\|^2$

Subject to: $\tilde{\mathbf{H}}\tilde{\mathbf{f}}_0 = \frac{1}{M}\Delta$.

The LS solution to this problem is

$$\tilde{\mathbf{f}}_0 = \frac{1}{M}\tilde{\mathbf{H}}^+\Delta \tag{43}$$

where

$$\tilde{\mathbf{H}}^+ = \tilde{\mathbf{H}}^* \left(\tilde{\mathbf{H}}\tilde{\mathbf{H}}^* \right)^{-1} \tag{44}$$

is the pseudoinverse of $\tilde{\mathbf{H}}$, $\det(\tilde{\mathbf{H}}\tilde{\mathbf{H}}^*) \neq 0$, and $\tilde{\mathbf{H}}^*$ is the complex conjugate transpose of $\tilde{\mathbf{H}}$. Given the solution in (43), the synthesis prototype filter coefficients are then given by

$$\mathbf{f}_0 = \mathbf{P}^{-1}\tilde{\mathbf{f}}_0. \tag{45}$$

D. Oversampled Case with Rational Oversampling Factor

The dimension of $\mathbf{H}$ given in (13) which is a function of N, M, and D determines whether the design problem is over-determined (no solution for $\mathbf{f}_0$) or under-determined (an infinite number of solutions for $\mathbf{f}_0$). Depending on the case, we may design $\mathbf{f}_0$ according to the previous cases.

V. Design Examples and Performance Analysis

The design algorithm for uniform-DFT linear phase filter banks is as follows.

1. Design an analysis prototype LPF, $\mathbf{h}_0$ with linear phase

2. (Critically sampled case) Determine $\bar{\mathbf{H}}_l$ for each l as in (17)

   (Oversampled case) Determine $\tilde{\mathbf{H}}$ as in (42)

3. (Critically sampled case) Compute $\bar{\mathbf{f}}_l$ for each l according to (20)

   (Oversampled case) Compute $\tilde{\mathbf{f}}_0$ according to (45)

4. The synthesis prototype LPF, $\mathbf{f}_0$ is constructed according to (18) or (45) for the critically sampled or oversampled case, respectively. $\mathbf{f}_0$ will be linear phase.

A MATLAB program which implements the above algorithm is available at [10]. We present several design examples illustrating the method.

Example 1 (Critically-sampled) Here we have $M = D = 4$ and the analysis prototype LPF is designed with a length $N = 32$, Hamming-windowed sinc function (`fir1` in MATLAB). Figs. 4 and 5 illustrate magnitude responses of the prototype filters. Fig. 6 illustrates the worst-case impulse response, i.e. the response to $\delta(n-l)$ that has the largest magnitude component (artifact) other than the main, delayed impulse. Fig. 7 illustrates the magnitude response of the filter bank for the worst-case impulse response.

Example 2 (Oversampled) Here we have $M = 4$. $D = 2$ and the analysis prototype LPF is designed with a length $N = 32$, Hamming-windowed sinc function (`fir1` in MATLAB). Figs. 4 and 8 illustrate the magnitude responses of prototype filters. Fig. 9 illustrates the impulse response, and Fig. 10 illustrates the magnitude response of the filter bank for the worst-case impulse response.

VI. Conclusions

This paper has presented a least-squares solution to the design of uniform-DFT linear phase filter banks. In the critically sampled case, the design yields near perfect reconstruction while in the oversampled case, the design yields perfect reconstruction. An algorithm and examples were presented

to demonstrate the method.

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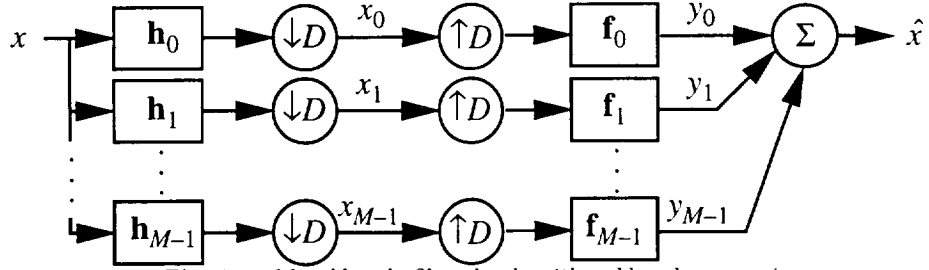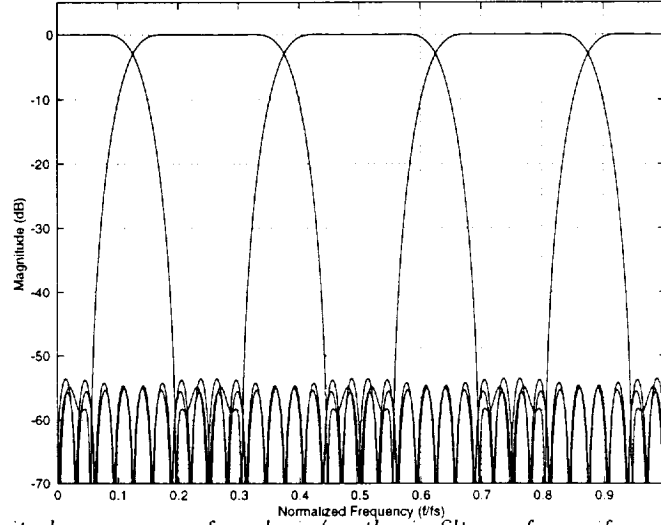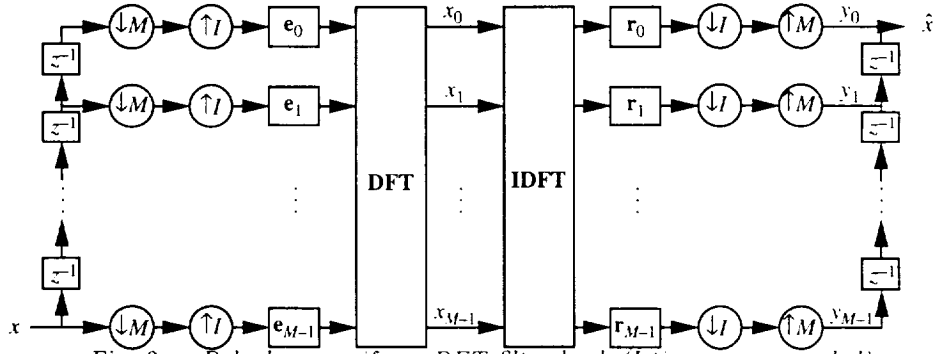
Fig. 1.   *M-subband, filter bank with subband processing*

Fig. 2.   *Magnitude responses of analysis/synthesis filters of a uniform-DFT filter bank*

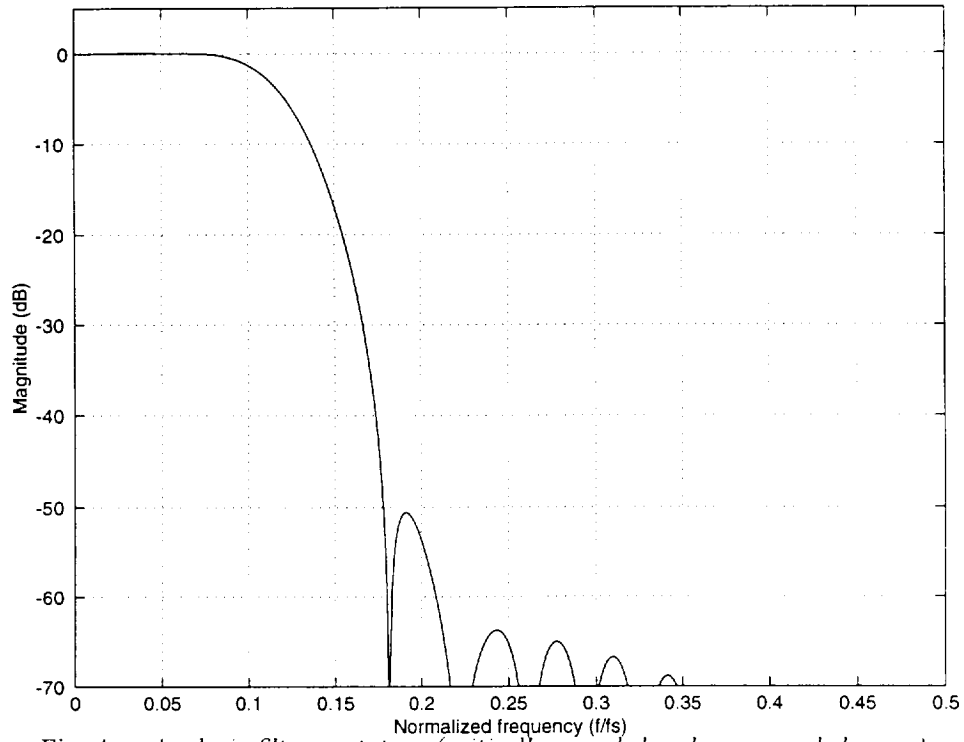Fig. 3.   *Polyphase, uniform-DFT filter bank (I-times oversampled)*

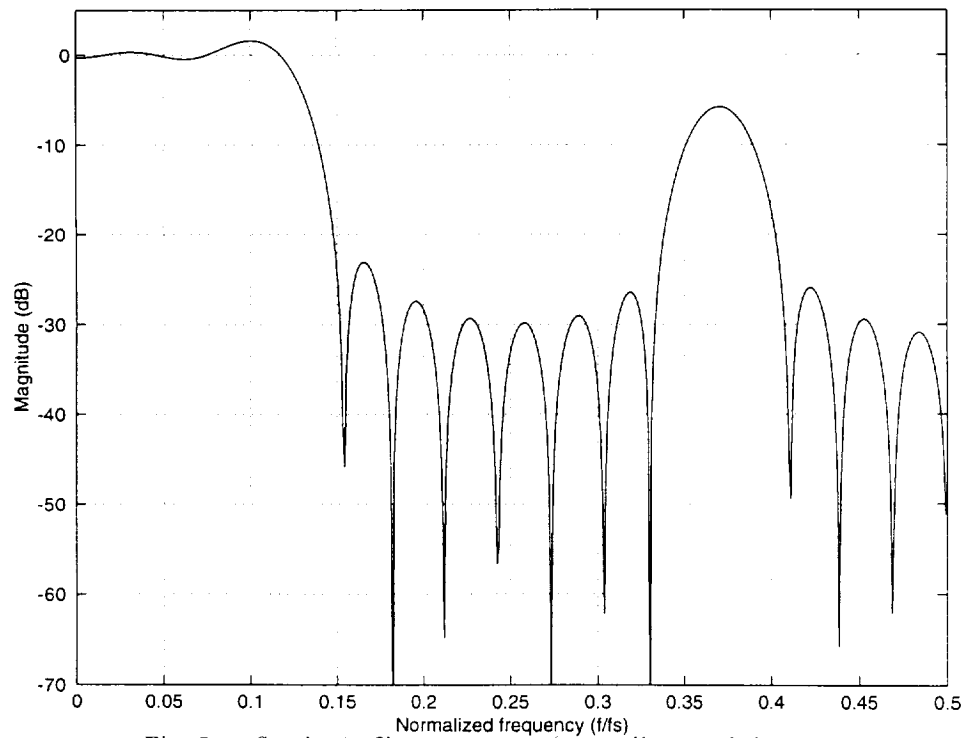Fig. 4. *Analysis filter prototype (critically-sampled and oversampled cases).*

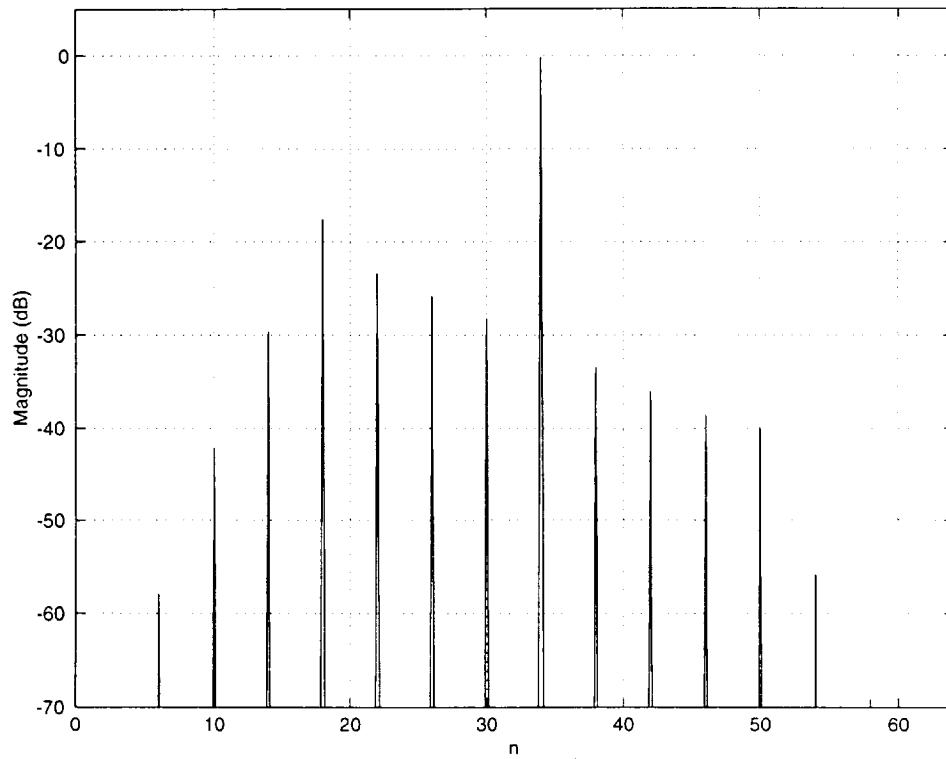Fig. 5. *Synthesis filter prototype (critically-sampled case).*

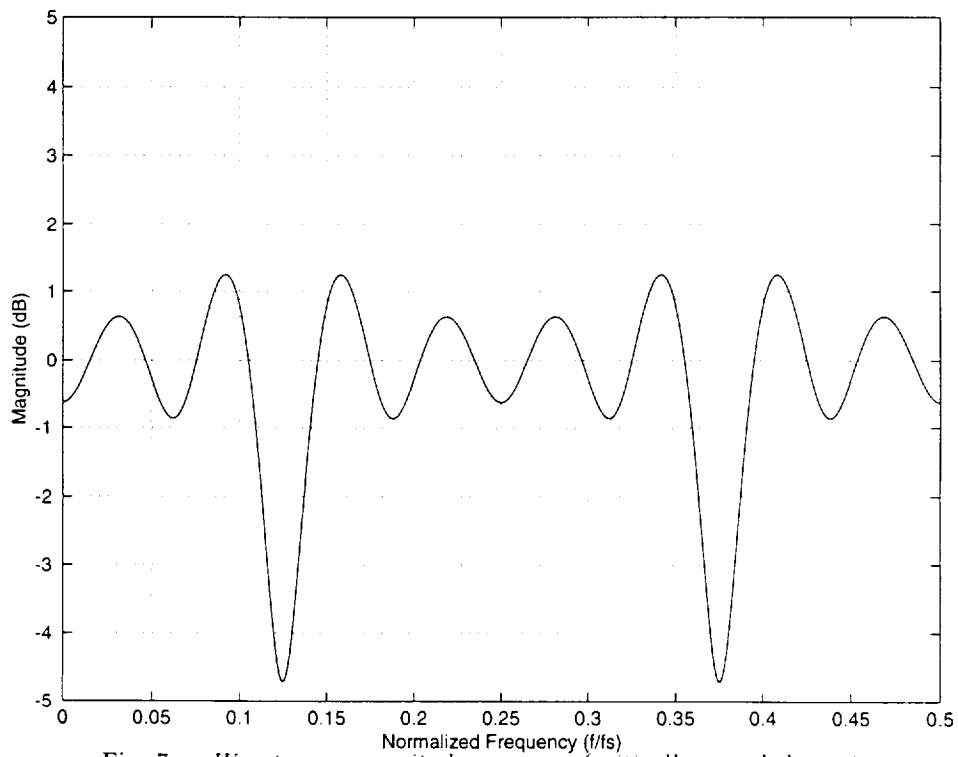Fig. 6. *Worst-case impulse response (critically-sampled case).*

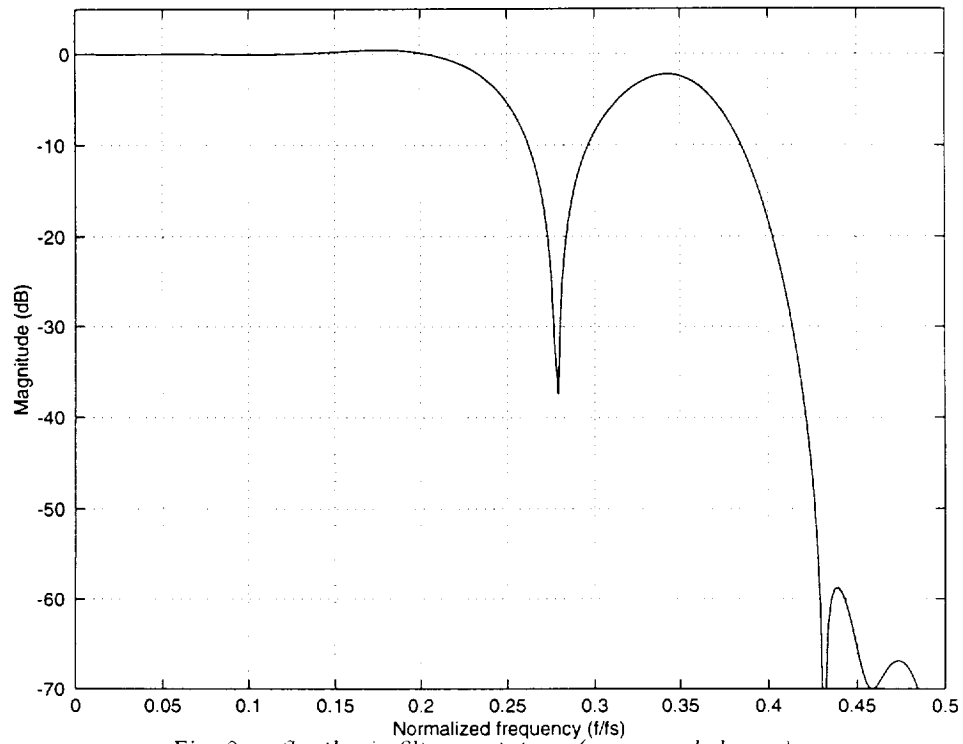Fig. 7. *Worst-case magnitude response (critically-sampled case).*

Fig. 8.   *Synthesis filter prototype (oversampled case).*
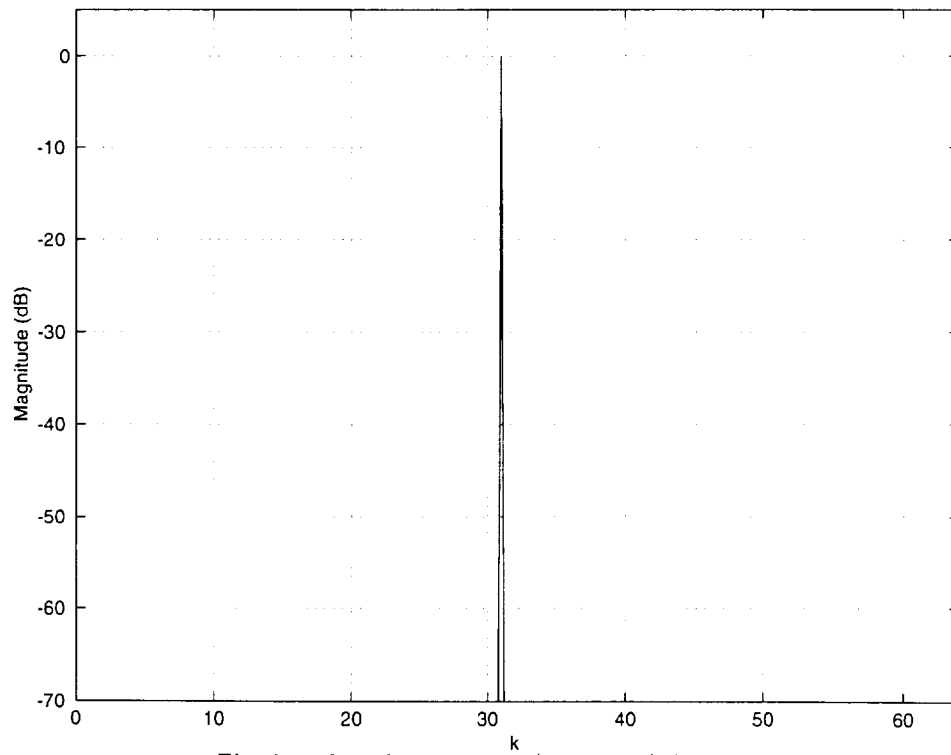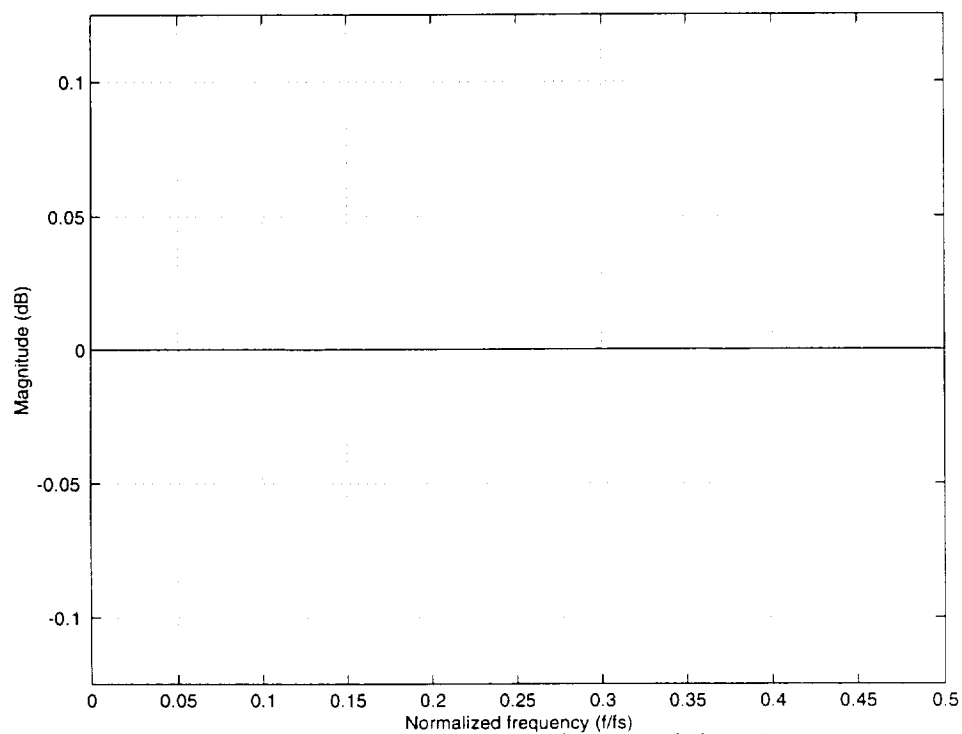

Fig. 9.   *Impulse response (oversampled case).*

Fig. 10.   *Magnitude response (oversampled case).*